

## TRANSVERSALS IN TREES

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A *transversal* in a rooted tree is any set of nodes that meets every path from the root to a leaf. We let  $c(T, k)$  denote the number of transversals of size  $k$  in a rooted tree  $T$ . If  $T$  has  $n$  nodes and  $n \geq 2$ , then

$$\begin{aligned} c(T, n) &= 1, \\ c(T, n-1) &= n, \end{aligned}$$

and

$$\binom{n-1}{k-1} \leq c(T, k) \leq \binom{n}{k} \quad \text{for all } k = 1, 2, \dots, n-2. \quad (1)$$

The  $n-2$  upper bounds in (1) are attained simultaneously if and only if  $T$  has precisely one leaf; the  $n-2$  lower bounds in (1) are attained simultaneously if and only if  $T$  has precisely  $n-1$  leaves, in which case the root has  $n-1$  children. How high can these lower bounds be raised if an upper bound smaller than  $n-1$  is imposed on the number of children of every node of  $T$ ? This is the question we are going to answer. Its creative interpretation has appeared in [1].

Harary and Schwenk [3] call a tree (an unrooted one) where removal of all vertices of degree one produces a path a *caterpillar*. Abusing this usage a little, we will call a caterpillar any rooted tree where removal of all leaves produces a rooted tree with precisely one leaf. By a *full caterpillar of degree  $d$* , we will mean a caterpillar where each internal node, except possibly the lowest one, has precisely  $d$  children.

**Theorem 1** *Let  $n$  and  $d$  be positive integers such that  $d < n$ ; let  $T$  be any rooted tree on  $n$  nodes where each internal node has at most  $d$  children and let  $T^*$  be the full caterpillar of degree  $d$  on  $n$  nodes. Then  $c(T, k) \geq c(T^*, k)$  for all  $k = 1, 2, \dots, n$ .*

In the special case where  $d = 2$ , inequalities  $c(T, k) \geq c(T^*, k)$  with (essentially)  $k = 1, 2, \dots, 1 + \lfloor \log_2 n \rfloor$  were proved in [2] by an argument different from the argument given below.

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Given rooted trees  $T, T'$  on  $n$  nodes, we will write  $T \succ T'$  to mean that  $c(T, k) \geq c(T', k)$  for all  $k = 1, 2, \dots, n$  and that  $c(T, k) > c(T', k)$  for at least one of these values of  $k$ . With this notation, a refinement of Theorem 1 can be stated as follows.

**Theorem 2** *Let  $n$  and  $d$  be positive integers such that  $d < n$  and let  $T^*$  be the full caterpillar of degree  $d$  on  $n$  nodes. If  $T$  is a rooted tree on  $n$  nodes such that each internal node of  $T$  has at most  $d$  children, then  $T \succ T^*$  or else  $T = T^*$ .*

Our proof of Theorem 2 relies on two ways of altering a rooted tree  $T$  so that the resulting tree succeeds  $T$  in the partial order  $\succ$ . We shall describe these alterations in terms of the *parent function* of a rooted tree that assigns to each node  $z$  of the tree its parent  $p(z)$  — except when  $z$  is the root, in which case  $p(z)$  is undefined.

**Lemma 1** *Let  $T$  be a rooted tree defined by parent function  $p$ . Let  $x$  and  $y$  be nodes of  $T$  such that  $x$  is not the root and  $y$  is a proper ancestor of  $p(x)$ . Let  $T'$  be the rooted tree defined by parent function  $p'$  such that*

$$p'(z) = \begin{cases} p(z) & \text{if } z \neq x, \\ y & \text{if } z = x. \end{cases}$$

*Then  $T \succ T'$ .*

**Proof.** If  $z$  is a leaf of  $T$ , then  $z$  is a leaf of  $T'$  and every node on the path from the root to  $z$  in  $T'$  lies on the path from the root to  $z$  in  $T$ . It follows that every transversal in  $T'$  is a transversal in  $T$ , and so  $c(T, k) \geq c(T', k)$  for all  $k$ . To see that  $c(T, k) > c(T', k)$  for at least one  $k$ , consider the set that consists of  $p(x)$  and all leaves of  $T$  that are not descendants of  $p(x)$ : this set is a transversal in  $T$  but not in  $T'$ . QED

**Lemma 2** *Let  $T$  be a rooted tree defined by parent function  $p$ . Let  $x$  and  $y$  be nodes of  $T$  such that  $x$  is not the root,  $x$  is not a leaf, and  $y$  is a leaf which is a proper descendant of a sibling of  $x$ . Let  $T'$  be the rooted tree defined by parent function  $p'$  such that*

$$p'(z) = \begin{cases} p(z) & \text{if } p(z) \neq x, \\ y & \text{if } p(z) = x. \end{cases}$$

*Then  $T \succ T'$ .*

**Proof.** Given any set  $S'$  of nodes in  $T'$ , define

$$f(S') = \begin{cases} S' & \text{if } S' \text{ meets the path from the root to } y, \\ S' - \{x\} \cup \{y\} & \text{otherwise.} \end{cases}$$

If  $S'$  is a transversal in  $T'$ , then  $f(S')$  is a transversal in  $T$  and  $|f(S')| = |S'|$ . If  $f(S_1) = f(S_2)$  and  $S_1 \neq S_2$ , then at least one of  $S_1, S_2$  avoids the path from the root to  $x$ , and so it is not a transversal in  $T'$ . It follows that  $c(T, k) \geq c(T', k)$  for all  $k$ . To see that  $c(T, k) > c(T', k)$  for at

least one  $k$ , consider the set that consists of  $p(y)$  and all leaves of  $T$  that are not descendants of  $p(y)$ : this set  $S$  is a transversal in  $T$ , but there is no transversal  $S'$  in  $T'$  such that  $f(S') = S$ . QED

**Proof of Theorem 2.** Consider any rooted tree  $T$  on  $n$  nodes such that each internal node of  $T$  has at most  $d$  children. Assuming that there is no rooted tree  $T'$  on  $n$  nodes such that each internal node of  $T'$  has at most  $d$  children and such that  $T \succ T'$ , we shall prove that  $T$  is the full caterpillar of degree  $d$ . Lemma 2 guarantees that no two internal nodes of  $T$  are siblings, which means that  $T$  is a caterpillar; in turn, Lemma 1 guarantees that each internal node of  $T$ , except possibly the lowest one, has precisely  $d$  children. QED

Imposing an upper bound on the number of children of every node is a way of staying clear of the tree that attains simultaneously the  $n - 2$  lower bounds in (1), one where all children of the root are leaves. Another way to stay clear of this tree is to impose an upper bound on the number of leaves. There is a corresponding analogue of Theorem 2 and this analogue also follows directly from our two lemmas.

**Theorem 3** *Let  $n$  and  $m$  be positive integers such that  $m < n$  and let  $T^*$  be the caterpillar on  $n$  nodes, where the root has  $m$  children and every node other than the root has at most one child. If  $T$  is a rooted tree with  $n$  nodes and at most  $m$  leaves, then  $T \succ T^*$  or else  $T = T^*$ .*

**Proof.** Consider any rooted tree  $T$  with  $n$  nodes and at most  $m$  leaves. Assuming that there is no rooted tree  $T'$  with  $n$  nodes and at most  $m$  leaves such that  $T \succ T'$ , we shall prove that  $T = T^*$ . Lemma 2 guarantees that no two internal nodes of  $T$  are siblings, which means that  $T$  is a caterpillar; in turn, Lemma 1 guarantees that no internal node of  $T$  other than the root has two or more children. QED

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## References

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